Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

PERIODICALLY CLOSED CHAINS OF REDUCED FRACTIONS.

By A. Arwin.

Introduction. In § 1 we first have to prove the existence of the positively and in § 5 of the negatively reduced fractions. With integers P_ν, Q_ν and D, we call the element $(P_\nu + \sqrt{D})/Q_\nu$ a positively reduced fraction, if the inequalities $0 < (P_\nu - \sqrt{D})/Q_\nu < 1$, $(P_\nu + \sqrt{D})/Q_\nu > 1$ are satisfied, and the element $(\sqrt{D} + P_\tau)/Q_\tau$ a negatively reduced fraction, if $0 < (\sqrt{D} - P_\tau)/Q_\tau < 1$, $(\sqrt{D} + P_\tau)/Q_\tau > 1$. After the positively reduced fractions in § 1 have been defined, their chain formations will be studied, that is the formation

$$(1) \qquad \frac{P_\rho + \sqrt{D}}{Q_\rho} = \beta_\rho - \frac{P_{\rho+1} - \sqrt{D}}{Q_\rho} = \beta_\rho - \frac{1}{\dfrac{P_{\rho+1} + \sqrt{D}}{Q_{\rho+1}}}, \text{ \&c.}$$

where β_ρ is always the integer next over the greatest integer in $(P_\rho + \sqrt{D})/Q_\rho$, and then we have to prove our fundamental theorem: *Whichever element opens a chain of the nature* (1), *we always end in a periodically closed chain of reduced fractions, and only a finite number of such chains exists.* Next we shall see that these chains include all possible positively reduced fractions. Moreover, the chain complementary to that derived from $(P_\nu + \sqrt{D})/Q_\nu$ as its first element being by definition the chain derived from the first element $(P_\nu + \sqrt{D})/Q_{\nu-1}$ where $P_\nu^2 - D = Q_{\nu-1} Q_\nu$, it will be shown that the complementary chain is periodic and the inverse of the original chain. These two chains may be different, or they may coincide. In the latter case we shall prove that the continued fraction is symmetrical. The different types of symmetry will be given, and the reason for their symmetry will be shown. At the end of § 1 another fundamental formula, valid for all formations of fractions, will be proved, and it will also be shown how the closed chains give solutions to Pell's equation. In § 2 attention is called to some types of remarkably regular formations of chains, so that we may survey them better. In § 3 we prove the following theorem: The necessary and sufficient condition that the indeterminate equation $x^2 - Dy^2 = A$ have a solution is that the chain formation (1) shall run into the special, periodic chain that contains in it $Q_0 = 1$, when all solutions may easily be taken from the related continued fraction. At the same time we discuss the problem of writing a given number A in the form (a, b, c), that is of solving in integers x and y the equation $ax^2 + 2bxy + cy^2 = A$. In § 4 the chain containing the element $Q_\nu = 1$ or $(P_\nu + \sqrt{D})/1$, the so called principal chain which customarily is used in solving the Pellian equation is investigated. In § 5 the negatively reduced fractions are introduced, their existence proved, also their connection with the positively reduced fractions. Then, proceeding from the results found, we shall see that Gauss's classical theory of binary quadratic forms, al least in its main characteristics, can be given by our theory of periodically closed chains of fractions, and in particular that the number of the positively closed chains is most intimately connected with the number of classes for a given determinant D, of quadratic forms and the negatively with the number of ideal classes in the domain $K(\sqrt{D})$. Finally in § 6 it will briefly be shown how quadratic forms with negative D can also be brought under this generalized theory of continued fractions.

§ 1.

We shall now prove the existence of positively reduced fractions $(P_\nu + \sqrt{D})/Q_\nu$, defined by the two inequalities $0 < (P_\nu - \sqrt{D})/Q_\nu < 1$, $(P_\nu + \sqrt{D})/Q_\nu > 1$.

THEOREM 1: *The number of the periodically closed chains is finite.* We form the chain of values

$$(2') \qquad P_0^2 - D = Q_1 Q_0, \; P_1^2 - D = Q_0 Q_1, \; \&c.$$

with $P_\tau + P'_{\tau+1} \equiv 0 \bmod Q_\tau$ and determine for $0 < (P'_{\nu+1} - \sqrt{D})/Q_\nu > 1$ the number n from $0 < (P'_{\nu+1} - nQ_\nu - \sqrt{D})/Q_\nu < 1$ to be $n = [(P'_{\nu+1} - \sqrt{D})/Q_\nu]$, where [] as usual is the greatest integer in $(P'_{\nu+1} - \sqrt{D})/Q_\nu$, and $P'_{\nu+1} - nQ_\nu = P_{\nu+1}$. Obviously we must have $P_{\nu+1} < Q_\nu$ for $Q_\nu > D$. For, $P_{\nu+1} - Q_\nu - \sqrt{D} > 0$, since $(P_{\nu+1} - Q_\nu)^2 - D = Q_\nu Q_{\nu+1} > 0$; and hence the inequalities $0 < (P_{\nu+1} - \sqrt{D})/Q_\nu < 1$ and $0 < (P_{\nu+1} - Q_\nu - \sqrt{D})/Q_\nu < 1$ are incompatible. If $P'^2_{\nu+1} - D < 0$ we take n from the inequality $0 < (P'_{\nu+1} + nQ_\nu - \sqrt{D})/Q_\nu$ and get $n = [(\sqrt{D} - P'_{\nu+1})/Q_\nu] + 1$. Therefore we may also write (2′) as the following chain of fractions

$$(2'') \qquad \begin{aligned} (P_0 + \sqrt{D})/Q_0 &= \beta_0 - (P_1 - \sqrt{D})/Q_0 = \beta_0 - \frac{1}{(P_1 + \sqrt{D})/Q_1}, \\ (P_1 + \sqrt{D})/Q_1 &= \beta_1 - (P_2 - \sqrt{D})/Q_1 = \beta_1 - \frac{1}{(P_2 + \sqrt{D})/Q_2}, \end{aligned}$$

in which, since $0 < (P_{\nu+1} - \sqrt{D})/Q_\nu < 1$, β_ν is the integer next above the greatest integer in $(P_\nu + \sqrt{D})/Q_\nu$. The chain formation (2″) is preferable to (2′). In the sequel we may therefore refer to the two chains (2′) and (2″) as both uniquely determined and identical with each other. Let us now suppose $Q_\tau > D$. Then we get $P^2_{\tau+1} - D = Q_\tau$. $Q_{\tau+1} < P^2_{\tau+1} < Q^2_\tau$, hence $Q_{\tau+1} < Q_\tau$ and thus as far as $Q_\tau > D$ the set of values $Q_\nu\,(\nu > \tau)$ succesively decreases until we get a $Q_\rho < D$ with a $P'_{\rho+1}$, that is taken as least positive remainder from $-P_\rho \equiv P'_{\rho+1}$ mod. Q_ρ giving $P'^2_{\rho+1} - D < 0$. If in this case we take n as above, we get the following inequalities $D + \sqrt{D} > P'_{\rho+1} + nQ_\rho > \sqrt{D}$ because of the inequality $n = (\sqrt{D} - P'_{\rho+1})/Q_\rho - \mu + 1 < (D + \sqrt{D} - P'_{\rho+1})/Q_\rho$, which in its turn is always satisfied for values $\mu < 1$, $Q_\rho < D$. With regard to $(P'_{\rho+1} + nQ_\rho)^2 - D < (D + \sqrt{D})^2 - D = D(D + 2\sqrt{D}) = M(D)$ we must in $(P'_{\rho+1} + nQ_\rho)^2 - D = P^2_{\rho+1} - D = Q_\rho Q_{\rho+1}$ have $Q_{\rho+1} < M(D)$. When $Q_{\rho+1} > D$ the $Q_\tau\,(\tau > \varrho + 1)$, as we have shown, may be reduced so as to be less than D. In the cases $Q_{\rho+1} < D$, $P'^2_{\rho+2} - D < 0$ the above process upon repetition gives $Q_{\rho+2} < M(D)$. We now have $P_{\rho+1} = P'_{\rho+1} + nQ_\rho < D + \sqrt{D} < M(D)$ and when $Q_{\rho+1} > D$ also $P_{\rho+2} < Q_{\rho+1} < M(D)$. Hence in every case the set of values P_ν, Q_ν in (2″) can be reduced permanently below the limit $M(D)$. As however we may form fractions indefinitely, we must in every chain get the same set of fractions periodically repeated and must also have the number of existing chains finite, and thus our first fundamental theorem is proved. We shall now prove

THEOREM 2. *The fractions in the periodically closed chains are positively reduced.*

We have first to prove that, given the chain

$$\begin{aligned} &\cdots\cdots\cdots \\ &P^2_{\nu-1} - D = Q_{\nu-2} Q_{\nu-1}, \\ &P^2_\nu - D = Q_{\nu-1} Q_\nu, \\ &P^2_{\nu+1} - D = Q_\nu Q_{\nu+1}, \\ &\cdots\cdots\cdots \end{aligned} \tag{2'}$$

and for example $0 < (P_\nu - \sqrt{D})/Q_\nu < 1$, $(P_\nu + \sqrt{D})/Q_\nu > 1$ all subsequent fractions satisfy precisely similar inequalities. From $(P_\nu + \sqrt{D})/Q_\nu = \beta_\nu - (P_{\nu+1} - \sqrt{D})/Q_\nu > 1$ we conclude $\beta_\nu \geq 2$. Hence $(P_{\nu+1} + \sqrt{D})/Q_\nu > 1$. Then $[(P_{\nu+1} + \sqrt{D})/Q_\nu]\,[(P_{\nu+1} - \sqrt{D})/Q_{\nu+1}] = 1$, $[(P_{\nu+1} - \sqrt{D})/Q_\nu]\,[(P_{\nu+1} + \sqrt{D})/Q_{\nu+1}] = 1$ will directly give us $0 < (P_{\nu+1} - \sqrt{D})/Q_{\nu+1} < 1$, $(P_{\nu+1} + \sqrt{D})/Q_{\nu+1} > 1$ and thus similar inequalities are valid for all fractions later in the chain. Further, from the law of formation in the chain, all elements

$(P_{\tau+1}-\sqrt{D})/Q_\tau$ from the second column of (2″) lie in the interval between 0 and 1. If therefore the element $(P_\nu+\sqrt{D})/Q_\nu$ is the opening element of a periodically closed chain in the first column, the conjugate of the complementary element $(P_\nu+\sqrt{D})/Q_{\nu-1}$ must be found in the second column of the last row in order to close the chain. From $P_\nu^2-D=Q_{\nu-1}Q_\nu>0$, $[(P_\nu-\sqrt{D})/Q_{\nu-1}]\,[(P_\nu+\sqrt{D})/Q_\nu]=1$ we have $(P_\nu+\sqrt{D})/Q_\nu>1$, and similarly for each element in the periodically closed chain. Now for any element $(P_\tau-\sqrt{D})/Q_\tau>0$ having $(P_\tau-\sqrt{D})/Q_\tau<1$ we have proved that all elements in the periodic chain must satisfy the same inequality. Let us therefore assume $(P_\tau-\sqrt{D})/Q_\tau>1$ for all τ. Then we must also for all τ have $Q_{\tau-1}>Q_\tau$. For if there were only one $\tau=\chi$ with $Q_{\chi-1}<Q_\chi$ we should have $P_\chi^2-D=Q_{\chi-1}Q_\chi<Q_\chi^2$, $[(P_\chi-\sqrt{D})/Q_\chi]\,[(P_\chi+\sqrt{D})/Q_\chi]<1$, contrary to $(P_\chi+\sqrt{D})/Q_\chi>1$, $(P_\chi-\sqrt{D})/Q_\chi>1$. But since for all τ in the complementary chain $P_\rho^2-D=Q_\rho Q_{\rho-1}$, &c. we have $Q_{\tau-1}>Q_\tau$, it would then follow that the chain could not be closed. Thus Theorem 2 is proved.

Let us assume the chain to be closed first with this row of elements

$$(3)\qquad (P_{\nu+\mu}+\sqrt{D})/Q_{\nu+\mu} = \beta_{\nu+\mu}-(P_{\nu+\mu+1}-\sqrt{D})/Q_{\nu+\mu}=\beta_{\nu+\mu}-\frac{1}{(P_{\nu+\mu+1}+\sqrt{D})/Q_{\nu+\mu+1}}.$$

The chain of the element $(P_{\nu+\mu}+\sqrt{D})/Q_{\nu+\mu}$ and also that of the element $(P_{\nu+\mu+1}+\sqrt{D})/Q_{\nu+\mu}$ is uniquely determined by the stated inequalities. If the periodic chain were now to be opened with the row in (3), the formation of chain with $(P_{\nu+\mu+1}+\sqrt{D})/Q_{\nu+\mu}$ might then not bring us out of the closed sets of values from the periodic part of the chain, but according to the given inequalities, assumed valid from the element $(P_\nu+\sqrt{D})/Q_\nu$, this same chain formation must restore the element $P_\nu-\sqrt{D})/Q_\nu$. Hence $(P_\nu+\sqrt{D})/Q_\nu$ itself opens the chain. As a corollary we have immediately: Where the positively reduced element $(P_\nu+\sqrt{D})/Q_\nu$ forms a continued fraction, the complementary element $(P_\nu+\sqrt{D})/Q_{\nu-1}$ will give the inverse continued fraction.

We shall next prove the important theorem:

THEOREM 4. *The reduced element* $(P_\tau+\sqrt{D})/Q_\tau$ *satisfies the inequalities*

$$P_\tau\leq D,\qquad Q_\tau\leq D.$$

Having $-P_\tau\equiv\alpha_\tau \bmod. Q_\tau$, $\alpha_\tau^2-D<0$ we always obtain $Q_\tau\leq D$ from $D-\alpha_\tau^2=Q_\tau M$. Now, having $P_{\tau+1}<Q_\tau$ and thus $Q_\tau>Q_{\tau+1}$ with $P_{\tau+1}^2-D>0$, we obtain from $(P_{\tau+1}+\sqrt{D})/Q_\tau>1$ $(P_{\tau+1}+\sqrt{D})/Q_\tau=\mu_\tau+(\sqrt{D}$

$-\beta_\tau)/Q_\tau$, $\mu_\tau \geq 1$ as the greatest integer in $(P_{\tau+1} + \sqrt{D})/Q_\tau$ and $\beta_\tau > 0$, for since $\beta_\tau = -\gamma_\tau$, $\gamma_\tau > 0$ the relation $P_{\tau+1} - \gamma_\tau = \mu_\tau Q_\tau$ is not possible with $P_{\tau+1} < Q_\tau$. From $\sqrt{D} - \beta_\tau > 0$ and $D - \beta_\tau^2 = Q_\tau N$ we once more conclude $Q_\tau \leq D$ for all these τ. It therefore remains only to examine such cases as $Q_{\tau+1} > Q_\tau$, which follows from $P_{\tau+1} = \alpha_\tau + n Q_\tau$, as well as the case $P_{\tau+1} > Q_\tau$, $Q_\tau > Q_{\tau+1}$. When $P_{\tau+1}^2 - D = Q_\tau Q_{\tau+1}$ we have just seen that $Q_\tau \leq D$, and either

$$-P_{\tau+1} \equiv \alpha_{\tau+1} \,\mathrm{mod.}\, Q_{\tau+1}, \qquad \alpha_{\tau+1}^2 - D < 0,$$

whence $Q_{\tau+1} \leq D$, or

$$-P_{\tau+1} \equiv P_{\tau+2} \,\mathrm{mod.}\, Q_{\tau+1}, \qquad P_{\tau+2}^2 - D > 0.$$

In this, since $\beta_{\tau+1} \geq 2$, we must have $P_{\tau+1} + P_{\tau+2} \geq 2\, Q_{\tau+1}$, $P_{\tau+1} > Q_{\tau+1} > P_{\tau+2}$ and get as before $Q_{\tau+1} \leq D$. Since now in all cases $Q_\tau \leq D$, $Q_{\tau+1} \leq D$, we infer from $P_{\tau+1}^2 - D \leq D \cdot D$ that $P_{\tau+1}^2 \leq D^2$, and hence that $P_\rho \leq D$, $Q_\rho \leq D$, for all values of ϱ, which establishes theorem 4.

As example of periodically closed chain we give

$$\frac{11+\sqrt{79}}{14} = 2 - \frac{17-\sqrt{79}}{14} = 2 - \cfrac{1}{\cfrac{17+\sqrt{79}}{15}},$$

$$\frac{11+\sqrt{79}}{7} = 3 - \frac{10-\sqrt{79}}{7} = 3 - \cfrac{1}{\cfrac{10+\sqrt{79}}{3}},$$

$$\frac{17+\sqrt{79}}{15} = 2 - \frac{13-\sqrt{79}}{15} = 2 - \cfrac{1}{\cfrac{13+\sqrt{79}}{6}},$$

$$\frac{10+\sqrt{79}}{3} = 7 - \frac{11-\sqrt{79}}{3} = 7 - \cfrac{1}{\cfrac{11+\sqrt{79}}{14}},$$

$$\frac{13+\sqrt{79}}{6} = 4 - \frac{11-\sqrt{79}}{6} = 4 - \cfrac{1}{\cfrac{11+\sqrt{79}}{7}}.$$

One of the following cases is to be expected: Either the two chains, the original and the complementary, are distinct or coincident. We now discuss the latter case. Given that $(P_{\tau+1} + \sqrt{D})/Q_{\tau+1}$ follows $(P_\tau + \sqrt{D})/Q_\tau$, we ask in what order are the elements $(P_\tau + \sqrt{D})/Q_{\tau-1}$, $(P_{\tau+1} + \sqrt{D})/Q_\tau$? Obviously the elements in the complementary chain are to be taken as the conjugates of the elements in the second column and read from below. In the original chain the order therefore must be $(P_{\tau+1}+\sqrt{D})/Q_\tau$ to $(P_\tau+\sqrt{D})/Q_{\tau+1}$ from above. By simple transposition of the last row of elements to the initial part of the periodic chain we can then always arrange these chains to be symmetrical. About the middle of any chain the elements $(P_\rho+\sqrt{D})/Q_\rho$, $(P_\rho+\sqrt{D})/Q_{\rho-1}$ must either be successive or situated with specific properties on opposite sides of the element $(P_{\rho+1}+\sqrt{D})/Q_{\rho+1}$, according to the type of symmetry involved. In the former case the "middle element" is characterized by the relation $P_\nu = P_{\nu+1}$. As, further, we have throughout $P_\rho + P_{\rho+1} = \beta_\rho Q_\rho$ and for this middle element $P_{\nu+1} = P_\nu$, there results in this case the important equality

$$2P_\nu = \beta_\nu Q_\nu. \tag{5}$$

In the latter case we have

$$\begin{array}{rl} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \downarrow & P_\rho^2 - D = Q_{\rho-1}Q_\rho \;\;= Q_{\rho-1}Q_\rho, \\ & P_{\rho+1}^2 - D = Q_\rho Q_{\rho+1} \;\;= Q_\rho Q_\rho, \\ & P_\rho^2 - D = Q_{\rho+1}Q_{\rho+2} = Q_\rho Q_{\rho-1}, \;\uparrow \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array} \tag{6}$$

and the middle element characterized by $Q_{\nu+1} = Q_\nu$. Hence

$$D = P_{\nu+1}^2 - Q_{\nu+1}^2. \tag{7}$$

Calling the middle element in the former case an element of the first category and in the latter an element of the second category we may formulate the following theorem.

THEOREM 5. *The periodic chain being opened with an element of the second category* ($D = P_0^2 - Q_0^2$) *and having* $Q_1 \neq Q_0$, *the continued fraction formed will be symmetric throughout, otherwise the symmetry will only appear after* β_0, *the first coefficient in the continued fraction.*

Arranging the chain from the middle element as an initial element we find that the initial element, now the middle element, must have just the same properties as any middle element, and therefore in a symmetrical chain there must exist such two specific elements belonging to one of the categories mentioned. And further, in the first case the initial element of the first column and the last element of the second will be conjugate and therefore after the first element the complementary and the original coincide, and the symmetry is complete. In the latter case the element in the second row of the original chain and the last element from the second column are the first to be conjugate, giving symmetry only after β_0.

THEOREM 6. *If in the symmetrical chain one element of each category exists the period of the continued fraction formed is odd, if there are two elements of the same category the period is even.*

If the initial element and also the middle element are of the second category, the symmetry is complete and in the middle of the chain we find the elements $(P_\nu+\sqrt{D})/Q_\nu$ and $(P_{\nu+1}+\sqrt{D})/Q_\nu$ with the same denominators following each other. Hence the period must be even. And if the initial and also the middle element are of the first category, elements with the same denominators will group themselves symmetrically around $(P_{\nu+1}+\sqrt{D})/Q_{\nu+1}$, but as in this case symmetry appears only after β_0, the period must be even. In the same manner we have for symmetrical chains containing one element of either category an odd period, and the theorem is proved. Thus we may for the different cases set up the following scheme.

Period $= 2\nu$, $Q_1 \neq Q_0$ and $Q_{2\nu-1} \neq Q_{2\nu}$,

$$Q_{\nu-\tau} = Q_{\nu+\tau}, \qquad (\tau = 0, 1 \cdots \nu)$$

Period $= 2\nu$, $Q_1 \neq Q_0$ and $Q_{2\nu-1} = Q_{2\nu}$, $D = P^2_{2\nu} - Q^2_{2\nu} = P^2_\nu - Q^2_\nu$,

$$Q_{\nu-\tau-1} = Q_{\nu+\tau}, \qquad (\tau = 0, 1 \cdots \nu-1)$$

Period $= 2\nu+1$, $Q_1 \neq Q_0$ and $Q_{2\nu} \neq Q_{2\nu+1}$, $D = P^2_{\nu+1} - Q^2_{\nu+1}$,

$$Q_{\nu-\tau} = Q_{\nu+\tau+1}, \qquad (\tau = 0, 1 \cdots \nu)$$

Period $= 2\nu+1$, $Q_1 \neq Q_0$ and $Q_{2\nu} = Q_{2\nu+1}$,

$$Q_{\nu-\tau} = Q_{\nu+\tau}, \qquad (\tau = 0, 1 \cdots \nu)$$

Assuming now that we have an arbitrarily orientated, not necessarily closed, chain with the initial element $\chi_0 = (P_0+\sqrt{D})/Q_0$ and $\chi_\tau = (P_\tau+\sqrt{D})/Q_\tau$ standing anywhere in the chain, we may then formulate the following theorem.

THEOREM 7. *Having*

$$\frac{y_{\tau-1}}{z_{\tau-1}} = [\beta_0, \beta_1, \cdots \beta_{\tau-1}]$$

we may always take from the chain the relation

$$(Q_0 y_{\tau-1} - P_0 z_{\tau-1})^2 - D z^2_{2-1} = Q_0 Q_\tau.$$

Because[*]

$$(8') \qquad \chi_0 = \frac{\chi_\tau y_{\tau-1} - y_{\tau-2}}{\chi_\tau z_{\tau-1} - z_{\tau-2}}, \text{ and } \frac{y_{\tau-2}}{z_{\tau-2}} = [\beta_0, \beta_1, \cdots \beta_{\tau-2}],$$

we get

$$[(P_0 + \sqrt{D})/Q_0]\,[(P_\tau + \sqrt{D})/Q_\tau]\, z_{\tau-1} - [(P_0 + \sqrt{D})/Q_0]\, z_{\tau-2} = [(P_\tau + \sqrt{D})/Q_\tau]\, y_{\tau-1} - y_{\tau-2},$$

giving the two equations

$$(8'') \qquad \begin{aligned} (P_0 P_\tau + D)\, z_{\tau-1} - P_\tau Q_0 y_{\tau-1} - P_0 Q_\tau z_{\tau-2} &= -\, Q_0 Q_\tau y_{\tau-2}, \\ (P_0 + P_\tau) z_{\tau-1} - Q_0 y_{\tau-1} &= Q_\tau z_{\tau-2}. \end{aligned}$$

Hence

$$(9) \qquad \begin{aligned} (P_0^2 - D) z_{\tau-1} - Q_0 (P_0 - P_\tau) y_{\tau-1} &= Q_0 Q_\tau y_{\tau-2}, \\ Q_0 (P_0 + P_\tau) z_{\tau-1} - Q_0^2 y_{\tau-1} &= Q_0 Q_\tau z_{\tau-2}. \end{aligned}$$

According to the well-known relation $y_{\tau-1} z_{\tau-2} - y_{\tau-2} z_{\tau-1} = 1$[†], we have just the formula

$$(10) \qquad (Q_0 y_{\tau-1} - P_0 z_{\tau-1})^2 - D z_{\tau-1}^2 = Q_0 Q_\tau.$$

Calling the period χ we have, from $Q_\tau = Q_\chi$, $Q_0 = Q_\chi$ and $P_0 = P_\chi$,

$$(11) \qquad (Q_\chi y_{\chi-1} - P_\chi z_{\chi-1})^2 - D z_{\chi-1}^2 = Q_\chi^2.$$

Further we obtain from (8″) $2 P\chi z_{\chi-1} \equiv 0 \text{ mod. } Q\chi$ and, since D is without square factors, we get from (11) a solution of

$$x^2 - D y^2 = (2^\epsilon)^2. \qquad (\epsilon = 0 \text{ or } 1)$$

where, in the case when $D \equiv 5$ mod. 8 and Q_χ is even, ϵ may be equal to unity. Finally we remark that for $\tau = \nu$ and on account of (5) the elements of the first category also are charactrized by

$$(12) \qquad D \equiv 0 \text{ mod. } \frac{Q_\nu}{2^\epsilon}. \qquad (\epsilon = 0 \text{ or } 1)$$

---

[*] Perron, O.: Die Lehre von den Kettenbrüchen, p. 7.

[†] Perron, O., loc. cit., p. 16.

§ 2.

Referring to the formula (7) we are now able to construct systematically all periodically closed, symmetrical chains of fractions for a given D. We have therefore to determine P_0 and Q_0 according to

$$D = \mu\nu,\ Q_0 = 2^\varepsilon \mu,\ P_0 = \alpha\mu. \qquad (\varepsilon = 0 \text{ or } 1) \tag{13}$$

and to the inequalities $\alpha\mu > \sqrt{D}$, $(\alpha\mu - \sqrt{D})/\mu < 1$, hence α so that $\alpha\mu > \sqrt{D} > (\alpha - 1)\mu$ or

$$\alpha = \left[\frac{\sqrt{D}}{\mu}\right] + 1. \tag{14}$$

We must then prove $(\alpha\mu + \sqrt{D})/\mu$ to be a reduced fraction. But this is evident from $(\alpha\mu + \sqrt{D})/\mu > 1$. The possibility $\varepsilon = 1$ must always be discussed independently, giving $(\alpha\mu)^2 - \mu\nu = 2\mu\ [(\alpha^2\mu - \nu)/2]$, $0 < (\alpha\mu - \sqrt{D})/2\mu < 1$, $(\alpha\mu + \sqrt{D})/2\mu > 1$.

Since all chains with the initial element (13) have been constructed, the possible chain with $D = P_0^2 - Q_0^2$ and, as $\chi = 2\nu$, with $D = P_\nu^2 - Q_\nu^2$ are to be sought, which will completely determine all possible symmetric chains of reduced fractions. D being without square factors, $D = \mu\nu$ $(\mu > \nu)$ implies $P_0 = (\mu + \nu)/2$, $Q_0 = (\mu - \nu)/2$ with μ and ν odd, and we have also

$$0 < \frac{\dfrac{\mu+\nu}{2} - \sqrt{D}}{\dfrac{\mu-\nu}{2}} < 1, \qquad \frac{\dfrac{\mu+\nu}{2} + \sqrt{D}}{\dfrac{\mu-\nu}{2}} > 1,$$

since $\mu + \nu - 2\sqrt{\mu\nu} < \mu - \nu$, $\mu > \nu$. Thus $(P_0 + \sqrt{D})/Q_0$ is a reduced fraction generating a periodically closed chain. Returning to (10) and (13) we may conclude from $(Q_0 y_{\rho-1} - P_0 z_{\rho-1})^2 - D z_{\rho-1}^2 = Q_0 Q_\rho$ that these symmetrical chains yield solutions of any equation $\mu x^2 - \nu y^2 = 2^\varepsilon Q_\rho$, because of the relation

$$\mu\,(2^\varepsilon y_{\rho-1} - \alpha z_{\rho-1})^2 - \nu z_{\rho-1}^2 = 2^\varepsilon Q_\rho, \tag{15}$$

which is valid for all denominators Q_ρ of the reduced fractions.

We shall now endeavour to determine the system of integers which form the periodic chain in the particular case $D = \mu\nu$, $P_0 = Q_0 = \mu > \nu$ or $D = \mu\nu$, $P_\nu = Q_\nu = \mu$. In this case the following reduced chain subsists

$$\begin{gathered}
\mu^2-\mu\nu=\mu(\mu-\nu),\\
(\mu-2\nu)^2-\mu\nu=(\mu-\nu)(\mu-2^2\nu),\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
[\mu-n(n-1)\nu]^2-\mu\nu=[\mu-(n-1)^2\nu][\mu-n^2\nu],
\end{gathered}\tag{16'}$$

as far as $\mu-n^2\nu>0$, as may be checked by

$$0<\frac{\mu-n(n-1)\nu-\sqrt{D}}{\mu-n^2\nu}<1,\qquad\frac{\mu-n(n-1)\nu+\sqrt{D}}{\mu-n^2\nu}>1,$$

remembering that $n\sqrt{\nu}<\sqrt{\mu}$. From the identity

$$[\mu-n(n-1)\nu]+[\mu-n(n+1)\nu]=2[\mu-n^2\nu]$$

we infer that all β_τ equal 2 as far as this formation holds. Also in the general case $\alpha \neq 1$ we can construct such sets

$$\begin{gathered}
(\alpha\mu)^2-\mu\nu=\mu(\alpha^2\mu-\nu),\\
[\alpha(2\alpha-1)\mu-2\nu]^2-\mu\nu=(\alpha^2\mu-1^2\nu)((2\alpha-1)^2\mu-2^2\nu),\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\{[\overline{n-1}\cdot\alpha-\overline{n-2}][n\alpha-\overline{n-1}]\mu-n(n-1)\nu\}^2-\mu\nu\\
=\{[\overline{n-1}\cdot\alpha-\overline{n-2}]^2\mu-(n-1)^2\nu\}\{[n\alpha-\overline{n-1}]^2\mu-n^2\nu\},
\end{gathered}\tag{16''}$$

where also, from the identity

$$\begin{gathered}
\{[\overline{n-1}\cdot\alpha-\overline{n-2}][n\alpha-\overline{n-1}]\mu-n(n-1)\nu\}+\{[n\alpha-\overline{n-1}]\\
\times[\overline{n+1}\cdot\alpha-n]\mu-n(n+1)\nu\}=2\{[n\alpha-\overline{n-1}]^2\mu-n^2\nu\},
\end{gathered}$$

all β_τ are equal to 2. It is readily seen that, when $\alpha=1$, we recover from this (16'). Here also, when $[n\alpha-\overline{n-1}]\sqrt{\mu}-n\sqrt{\nu}>0$, the inequality may be tested. For we have

$$\begin{gathered}
[\overline{n-1}\cdot\alpha-\overline{n-2}][n\alpha-\overline{n-1}]\mu-n(n-1)\nu-\sqrt{\mu\nu}\\
=\{[n\alpha-\overline{n-1}]\sqrt{\mu}-n\sqrt{\nu}\}\{[\overline{n-1}\cdot\alpha-\overline{n-2}]\sqrt{\mu}+\overline{n-1}\cdot\sqrt{\nu}\}>0,
\end{gathered}$$

and from

$$\{[n\alpha-\overline{n-1}]\sqrt{\mu}-n\sqrt{\nu}\}\{(\alpha-1)\sqrt{\mu}+\sqrt{\nu}\}<0,$$

we have also

$$\{[n\alpha-\overline{n-1}]\sqrt{\mu}-n\sqrt{\nu}\}\,\{[\overline{n-1}\cdot\alpha-\overline{n-2}]\sqrt{\mu}+\overline{n-1}\cdot\sqrt{\nu}\}$$
$$<[n\alpha-\overline{n-1}]^2\mu-n^2\nu.$$

But again, since

$$\{[n\alpha-\overline{n-1}]\sqrt{\mu}+n\sqrt{\nu}\}\,\{[\overline{n-1}\cdot\alpha-\overline{n-2}]\sqrt{\mu}-\overline{n-1}\sqrt{\nu}\}$$
$$>[n\alpha-\overline{n-1}]^2\mu-n^2\nu,$$

it follows that

$$[n\alpha-\overline{n-1}]\,[\overline{n-1}\cdot\alpha-\overline{n-2}]\,\mu-n\,(n-1)\,\nu+\sqrt{\mu\nu}$$
$$>[n\alpha-\overline{n-1}]^2\mu-n^2\nu,$$

and this, when $\sqrt{\nu}>(\alpha-1)\sqrt{\mu}$, viz. $\sqrt{D}>(\alpha-1)\,\mu$ (which is always satisfied), is equivalent to

$$[\overline{n-1}\cdot\alpha-\overline{n-2}]\,\sqrt{\mu}-\overline{n-1}\cdot\sqrt{\nu}>[n\alpha-\overline{n-1}]\sqrt{\mu}-n\sqrt{\nu}\,.$$

For $D=\mu\nu=P_i^2-Q_i^2$ $(i=0,\ \nu\ \text{or}\ \nu+1)$, we have $P_i=(\mu+\nu)/2$, $Q_i=(\mu-\nu)/2$, and the reduced system

$$\begin{aligned}
&\left(\frac{\mu+\nu}{2}\right)^2-\mu\nu=\frac{\mu-\nu}{2}\,\frac{\mu-\nu}{2},\\
&\left(\frac{\mu-3\nu}{2}\right)^2-\mu\nu=\frac{\mu-1^2\nu}{2}\,\frac{\mu-3^2\nu}{2},\\
(17)\qquad&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\left(\frac{\mu-(2n-1)\,(2n+1)\,\nu}{2}\right)^2-\mu\nu=\frac{\mu-(2n-1)^2\nu}{2}\,\frac{\mu-(2n+1)^2\nu}{2}.
\end{aligned}$$

We find in fact that

$$\frac{(\sqrt{\mu}-\overline{2n+1}\sqrt{\nu})(\sqrt{\mu}+\overline{2n-1}\sqrt{\nu})}{(\sqrt{\mu}-\overline{2n+1}\sqrt{\nu})(\sqrt{\mu}+\overline{2n+1}\sqrt{\nu})}<1,$$

supposing $\sqrt{\mu}>\overline{2n+1}\cdot\sqrt{\nu}$, and

$$\frac{(\sqrt{\mu}+\overline{2n+1}\sqrt{\nu})(\sqrt{\mu}-\overline{2n-1}\sqrt{\nu})}{(\sqrt{\mu}+\overline{2n+1}\sqrt{\nu})(\sqrt{\mu}-\overline{2n+1}\sqrt{\nu})}>1.$$

Further, we find from

$$\frac{\mu-(2n-1)(2n+1)\nu}{2}+\frac{\mu-(2n+1)(2n+3)\nu}{2}=2\,\frac{\mu-(2n+1)^2\nu}{2},$$

that all β_τ are equal to 2 as far as this formation holds.

To illustrate the preceding we take $D = 105 = 3.5.7$ and consider first $Q_0 = 1$, whence, since $10 < \sqrt{105} < 11$, $P_0 = 11$, and then we have the following symmetrical chain of fractions

$$1)\qquad \begin{aligned}
\frac{11+\sqrt{105}}{1} &= 22 - \frac{11-\sqrt{105}}{1} = 22 - \cfrac{1}{\cfrac{11+\sqrt{105}}{16}},\\
\frac{21+\sqrt{105}}{21} &= 2 - \frac{21-\sqrt{105}}{21} = 2 - \cfrac{1}{\cfrac{21+\sqrt{105}}{16}},\\
\frac{11+\sqrt{105}}{16} &= 2 - \frac{21-\sqrt{105}}{16} = 2 - \cfrac{1}{\cfrac{21+\sqrt{105}}{21}},\\
\frac{21+\sqrt{105}}{16} &= 2 - \frac{11-\sqrt{105}}{16} = 2 - \cfrac{1}{\cfrac{11+\sqrt{105}}{1}}.
\end{aligned}$$

The period $\chi = 2\nu = 4$ and thus $P_2 = Q_2 = 21$ giving $D = 105 \equiv 0$ mod. 21. It may be remarked that the chain 1) is formed by the values of the system (16″). Further the continued fraction is, as it should be, symmetrical on both sides of $\beta_2 = 2$, $\beta_0 = 22$ excepted. This type of chain ($Q_0 = 1$), the principal chain, is commonly used to solve Pell's equation.

2) The case $\varepsilon = 1$ will give $Q_0 = 2$, $P_0 = 11$.

$$\begin{aligned}
\frac{11+\sqrt{105}}{2} &= 11 - \frac{11-\sqrt{105}}{2} = 11 - \cfrac{1}{\cfrac{11+\sqrt{105}}{8}},\\
\frac{13+\sqrt{105}}{8} &= 3 - \frac{11-\sqrt{105}}{8} = 3 - \cfrac{1}{\cfrac{11+\sqrt{105}}{2}},\\
\frac{11+\sqrt{105}}{8} &= 3 - \frac{13-\sqrt{105}}{8} = 3 - \cfrac{1}{\cfrac{13+\sqrt{105}}{8}}.
\end{aligned}$$

The period $\chi = 2\nu + 1 = 3$, $105 = P_2^2 - Q_2^2 = 13^2 - 8^2$ from $D = \mu\nu = 21.5$.

3) $Q_0 = 3$, $P_0 = 4.3$, found by $3\alpha > \sqrt{105} > 3(\alpha - 1)$.

$$\frac{12 + \sqrt{105}}{3} = 8 - \frac{12 - \sqrt{105}}{3} = 8 - \frac{1}{\frac{12 + \sqrt{105}}{13}},$$

$$\frac{14 + \sqrt{105}}{7} = 4 - \frac{14 - \sqrt{105}}{7} = 4 - \frac{1}{\frac{14 + \sqrt{105}}{13}},$$

$$\frac{12 + \sqrt{105}}{13} = 2 - \frac{14 - \sqrt{105}}{13} = 2 - \frac{1}{\frac{14 + \sqrt{105}}{7}},$$

$$\frac{14 + \sqrt{105}}{13} = 2 - \frac{12 - \sqrt{105}}{13} = 2 - \frac{1}{\frac{12 + \sqrt{105}}{3}}.$$

The period $2\nu = 4$, $Q_2 = 7$, $P_2 = 14$, $D = 105 \equiv 0 \bmod. 7$, and the chain is formed partly like (16″).

4) $Q_0 = 2.3$, $P_0 = 5.3$ by $3\alpha > \sqrt{105} > 3(\alpha - 2)$.

$$\frac{15 + \sqrt{105}}{6} = 5 - \frac{15 - \sqrt{105}}{6} = 5 - \frac{1}{\frac{15 + \sqrt{105}}{20}},$$

$$\frac{27 + \sqrt{105}}{24} = 2 - \frac{21 - \sqrt{105}}{24} = 2 - \frac{1}{\frac{21 + \sqrt{105}}{14}},$$

$$\frac{15 + \sqrt{105}}{20} = 2 - \frac{25 - \sqrt{105}}{20} = 2 - \frac{1}{\frac{25 + \sqrt{105}}{26}},$$

$$\frac{21 + \sqrt{105}}{14} = 3 - \frac{21 - \sqrt{105}}{14} = 3 - \frac{1}{\frac{21 + \sqrt{105}}{24}},$$

$$\frac{25 + \sqrt{105}}{26} = 2 - \frac{27 - \sqrt{105}}{26} = 2 - \frac{1}{\frac{27 + \sqrt{105}}{24}}, \text{ \&c.}$$

The period $\chi = 2\nu = 8$, $Q_4 = 2.7$, $P_4 = 3.7$, $D = 105 \equiv 0 \bmod. Q_4/2$.

5) $Q_0 = 5$, $P_0 = 3.55\alpha > \sqrt{105} > 5(\alpha - 1)$.

We find the period $\chi = 2\nu = 20$, $P_{10} = Q_{10} = 105$, $D = 105 \equiv 0 \bmod. 105$. We have here a very interesting example of the formation (16″) and at the same time of (16′), if we form the chain from the middle element $(105 + \sqrt{105})/105$ and this formation must hold as far as $105 - n^2 > 0$, thus up to $n = 10$. The last denominators of this formation (16′) for example are therefore to be calculated as 24 and 5 since $(105 - 9.10)^2 - 105 = (105 - 9^2)(105 - 10^2)$. The formation (16″) will extend as far as n determined from the inequality $[n\alpha - \overline{n-1}]\sqrt{\mu} - n\sqrt{\nu} > 0$, $(2n+1)^2\, 5 > 21n^2$, $20n + 5 > n^2$, thus also for $n = 20$ and therefore through the whole period.

6) $Q_0 = 2.5$, $P_0 = 15$.

We find the period $\chi = 2\nu + 1 = 11$, whence the middle element must be a fraction of the second category, in this case formed by $105 = [(105+1)/2]^2 - [(105-1)/2]^2$. Here we have a good example of the formation (17) in $\mu = 105$, $\nu = 1$, and it holds as far as $105 - (2n+1)^2 > 0$ or $n = 4$. Here also all β_τ are equal to 2.

The alternative $Q_0 = 7$, $P_0 = 14$ has been met in 3).

$Q_0 = 2{,}7$, $P_0 = 3.7$ on account of $7\alpha > \sqrt{105} > 7(\alpha - 2)$ have already been treated with the preceding values $Q_0 = 2.3$, $P_0 = 5.3$ in 4).

7) $Q_0 = 15$, $P_0 = 15$, since $\alpha = [105/15] + 1$.

7)
$$\frac{15+\sqrt{105}}{15} = 2 - \frac{15-\sqrt{105}}{15} = 2 - \frac{1}{\frac{15+\sqrt{105}}{8}},$$

$$\frac{17+\sqrt{105}}{23} = 2 - \frac{29-\sqrt{105}}{23} = 2 - \frac{1}{\frac{29+\sqrt{105}}{32}},$$

$$\frac{15+\sqrt{105}}{8} = 4 - \frac{17-\sqrt{105}}{8} = 4 - \frac{1}{\frac{17+\sqrt{105}}{23}},$$

$$\frac{29+\sqrt{105}}{32} = 2 - \frac{35-\sqrt{105}}{32} = 2 - \frac{1}{\frac{35+\sqrt{105}}{35}}, \text{ \&c.}$$

Period $\chi = 20 = 8$, $Q_4 = P_4 = 35$, $D = 105 \equiv 0 \bmod. 35$.

In the case $Q_0 = 2.15$ we cannot form any reduced fraction. $Q_0 = P_0 = 35$ has indeed been treated in 7), and when $Q_0 = 2.35$ no reduced fraction exists. $Q_0 = P_0 = 21$ has occurred in 1) and $Q_0 = 2.21$ is impossible. As for $Q_0 = P_0 = 105$ we have found this combination in 5). We have now in fact treated all cases $D = 105 \equiv 0 \bmod. Q_\nu/2^\varepsilon$ $(\varepsilon = 0{,}1)$, and it only remains to investigate whether any fractions of the second category may still exist. In this case we should have $D = P_0^2 - Q_0^2$, which, with $D = \mu\nu$, must give $P_0 = (\mu + \nu)/2$,

$Q_0 = (\mu - \nu)/2$, since D is taken without any square factors. In virtue of this rule we find with $\mu = 105$, $\nu = 1$, $105 = [(105+1)/2]^2 - [(105-1)/2]^2$, which appear in 6). With $\mu = 35$, $\nu = 3$ we have $105 = [(35+3)/2]^2 - [(35-3)/2]^2$ and hence the chain

$$8) \qquad \begin{aligned}
\frac{19+\sqrt{105}}{16} &= 2 - \frac{13-\sqrt{105}}{16} = 2 - \cfrac{1}{\cfrac{13+\sqrt{105}}{4}}, \\
\frac{11+\sqrt{105}}{4} &= 6 - \frac{13-\sqrt{105}}{4} = 6 - \cfrac{1}{\cfrac{13+\sqrt{105}}{16}}, \\
\frac{13+\sqrt{105}}{4} &= 6 - \frac{11-\sqrt{105}}{4} = 6 - \cfrac{1}{\cfrac{11+\sqrt{105}}{4}}, \\
\frac{13+\sqrt{105}}{16} &= 2 - \frac{19-\sqrt{105}}{16} = 2 - \cfrac{1}{\cfrac{19+\sqrt{105}}{16}}.
\end{aligned}$$

Period $2\nu = 4$, $D = 105 = P_2^2 - Q_2^2 = [(15+7)/2]^2 - [(15-7)/2]^2$. Obviously we must get a middle element of this same category, or this case would already have occurred. And thus all possibilities of a symmetrical chain for $D = 105$ are in fact constructed, since $105 = [(21+5)/2]^2 - [(21-5)/2]^2$ has appeared in 2). As in the case $D = 105$ there do not exist any other chains, all existing closed chains of fractions are therefore determined.

§ 3.

In this paragraph we shall prove the following theorem, which is fundamental to the problem of expressing a given number A in a given quadratic form (a, b, c).

THEOREM 8. *The necessary and sufficient condition that the equation*

$$(1) \qquad x^2 - Dy^2 = A,$$

shall have a solution for given A and $D > 0$, is that from any root $\alpha_1^{(1)}$ of $z^2 \equiv D$ mod. A *as initial element there will be generated the chain*

$$(2) \qquad \begin{aligned}
\alpha_1^{(1)2} - D &= A A_1, \\
\alpha_2^{(1)2} - D &= A_1 A_2, \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\alpha_{i-1}^{(1)2} - D &= A_{i-2} A_{i-1}, \\
\alpha_i^{(1)2} - D &= A_{i-1} \cdot 1.
\end{aligned}$$

All solutions belonging to the same congruence-root differ only by unit factors.

The first part of the theorem is obvious. For each chain (2) must run into a periodically closed chain; and by Theorem 7 we get a solution of (1) when, and only when, the chain is the principal chain, viz., when it contains the denominator $Q_0 = 1$. The second part we can prove as follows. We start from the pair of given relations

$$M_1^2 - DN_1^2 = A, \qquad M_2^2 - DN_2^2 = A,$$

where M_1, N_1 and M_2, N_2 are relatively prime, and both solutions belong to the same congruence-root $\alpha_1^{(1)}$. With $\alpha_0^{(1)} + \alpha_1^{(1)} = A$ we form

$$\frac{\alpha_0^{(1)} + \sqrt{D}}{A} = 1 - \frac{\alpha_1^{(1)} - \sqrt{D}}{A} = 1 - \frac{1}{\dfrac{\alpha_1^{(1)} + \sqrt{D}}{A_1}}, \text{ \&c.}$$

From (10) § 1 we get

$$(Ay_\rho - \alpha_0^{(1)} z_\rho)^2 - Dz_\rho^2 = 1 \cdot A,$$

and therefore, with $N_1 = z_\rho$, $M_1 = \pm(Ay_\rho - \alpha_0^{(1)} z_\rho)$, the congruence

(3) $\qquad M_1 \equiv \pm \alpha_1^{(1)} N_1 \bmod. A,$ and also $M_2 \equiv \pm \alpha_1^{(1)} N_2 \bmod. A.$

Forming the ratio

(4)
$$\begin{aligned} \frac{M_1 + N_1\sqrt{D}}{M_2 \pm N_2\sqrt{D}} &= \frac{(M_1 + N_1\sqrt{D})(M_2 \mp N_2\sqrt{D})}{A} \\ &= \frac{M_1 M_2 \mp DN_1N_2}{A} + \sqrt{D}\,\frac{M_2N_1 \mp M_1N_2}{A} = \beta + \gamma\sqrt{D}, \end{aligned}$$

we find by (3)

(5) $\qquad M_1 M_2 \equiv \pm \alpha_1^{(1)2} N_1 N_2, \qquad M_1 M_2 \equiv \pm D N_1 N_2 \bmod. A,$

and also

$$M_2 N_1 \equiv \pm M_1 N_2 \bmod. A,$$

whence $\beta + \gamma\sqrt{D}$ must be a unit, and Theorem 8 is proved. As a corollary we have the solution of

(6) $$ax^2 + 2bxy + cy^2 = A$$

in integers x and y, Gauss's well-known theorem. We form

(7) $$(ax + by)^2 - Dy^2 = aA$$

with any solution

(8) $$M_\nu^2 - DN_\nu^2 = aA$$

belonging to some congruence-root $z = \alpha_1^{(\nu)}$, $\alpha_1^{(\tau)}$ $(\tau = 1 \cdots \nu \cdots)$ being the roots of $z^2 \equiv D \bmod. a A$. Thus we have $\alpha_1^{(\tau)^2} \equiv D$, $b^2 \equiv D \bmod. a$ and at least for some one of $\alpha_1^{(\tau)}$ the congruence must hold, say $\alpha_1^{(\nu)}$, $\alpha_1^{(\nu)} \equiv b \bmod. a$. By (3) we get then either $M_\nu + b N_\nu \equiv 0$ or $M_\nu - b N_\nu \equiv 0 \bmod. a$. With $y = \pm N_\nu$ we must therefore always obtain x in $a x \pm b N_\nu = M_\nu$ as an integer, whence we must seek all solutions of (6) by forming the chains (2) starting from all congruence-roots $\alpha_1^{(i)}$ $(i = 1, 2 \cdots \tau) \bmod. a A$ such that

$$\alpha_1^{(\rho)} \equiv \pm b \bmod. a.$$

If A is a negative integer the chain terminates

$$\alpha_{i-1}^{(1)^2} - D = A_{i-2} A_{i-1}, \qquad \alpha_i^{(1)^2} - D = A_{i-1} A_i,$$

where $A_{i-1} = D - 1$ and $A_i = D$ by the properties of the closed chains of positively reduced fractions, given in the preceding paragraphs. As can be seen from (2), all A_τ in the chain may also be expressed in the principal form (1), and the integers x and y may be calculated by the continued fraction formed. But, conversely, since the solution

$$M_1^2 - D N_1^2 = A \tag{9'}$$

is given,

$$M_2^2 - D N_2^2 = A_1 \tag{9''}$$

may easily be computed. For, since $\alpha_1^{(1)2} - D = A A_1$ and since $M_1 \equiv \pm N_1 \alpha_1^{(1)}$, $M_1 \alpha_1^{(1)} \equiv \pm N_1 \alpha_1^{(1)2} \equiv \pm N_1 D \bmod. A$, the ratio of

$$\frac{(\alpha_1^{(1)} + \sqrt{D})(M_1 \mp N_1 \sqrt{D})}{A} = \frac{M_1 \alpha_1^{(1)} \mp N_1 D}{A} + \sqrt{D} \frac{M_1 \mp N_1 \alpha_1^{(1)}}{A}$$

will at once give us M_2 and N_2 as integers in (9″).

Example.

$$6x^2 - 20xy + 11y^2 = 99, \quad (6x - 10y)^2 - 34y^2 = 6.99.$$

We have mod. 594 the four roots ± 230, ± 122 and have $230 \equiv -10 \bmod. 6$ and also $122 \equiv -10 \bmod. 6$. Hence

$$230^2 - 34 = 594.89, \quad 37^2 - 34 = 89.15, \quad 8^2 - 34 = 15.2, \quad 6^2 - 34 = 2.1,$$

and, from

$$\frac{6+\sqrt{34}}{1} = 12 - \frac{6-\sqrt{34}}{1} = 12 - \frac{1}{\dfrac{6+\sqrt{34}}{2}},$$

$$\frac{8+\sqrt{34}}{15} = 3 - \frac{37-\sqrt{34}}{15} = 3 - \frac{1}{\dfrac{37+\sqrt{34}}{89}},$$

$$\frac{6+\sqrt{34}}{2} = 7 - \frac{8-\sqrt{34}}{2} = 7 - \frac{1}{\dfrac{8+\sqrt{34}}{15}},$$

$$\frac{37+\sqrt{34}}{89} = 3 - \frac{230-\sqrt{34}}{89} = 3 - \frac{1}{\dfrac{230+\sqrt{34}}{594}},$$

we calculate

$$12 - \frac{1|}{|2} - \frac{1|}{|3} - \frac{1|}{|3} = \frac{628}{53},$$

having

$$(628 - 6.53)^2 - 34.53^2 = 594.$$

Taking $y = 53$ we find $x = 140$ from equation $6x - 10.53 = 310$, but in view of $35^2 - 34.6^2 = 1$ we deduce the simpler solution $x = 2$, $y = 5$. Proceeding from the root 122 we form the chain

$$122^2 - 34 = 594.25,\ 28^2 - 34 = 25.30,\ 32^2 - 34 = 30.33,$$
$$34^2 - 34 = 33.34,$$

leading to

$$\frac{34+\sqrt{34}}{34} = 2 - \frac{34-\sqrt{34}}{34} = 2 - \frac{1}{\dfrac{34+\sqrt{34}}{33}},$$

$$\frac{32+\sqrt{34}}{30} = 2 - \frac{28-\sqrt{34}}{30} = 2 - \frac{1}{\dfrac{28+\sqrt{34}}{25}},$$

$$\frac{34+\sqrt{34}}{33} = 2 - \frac{32-\sqrt{34}}{33} = 2 - \frac{1}{\dfrac{32+\sqrt{34}}{30}},$$

$$\frac{28+\sqrt{34}}{25} = 6 - \frac{122+\sqrt{34}}{25} = 6 - \frac{1}{\dfrac{122+\sqrt{34}}{594}},$$

and have

$$2-\frac{1\,|}{|\,2}-\frac{1\,|}{|\,2}-\frac{1\,|}{|\,6}=\frac{21}{16},$$

giving the solution

$$34\,(21-16)^2-16^2=594.$$

Since $y=5$ and $6x-10.5=16$, we have $x=11$ as a solution of

$$6x^2-20xy+11y^2=-99$$

giving a check.

§ 4.

This paragraph will be devoted to the continued fractions, which appear in the principal chain $Q_0=1$. The main results may be summarized in the following theorem.

THEOREM 9. *In the principal chain having the period* $\chi=2\nu$ *the middle element is as usual characterized by*

$$(1)\qquad D\equiv 0 \qquad \text{mod. } Q_\nu/2^\varepsilon \qquad (\varepsilon=0,1)$$

and the fraction $y_{\nu-1}/z_{\nu-1}$ *leads to a solution* α_1, β_1 *of the equation*

$$(2)\qquad \mu x^2-\nu y^2=2^\varepsilon$$

with $D=\mu\nu$. *Then the solution* α_2, β_2 *of Pell's equation is given by means of equation*

$$(3)\qquad \alpha_2+\beta_2\sqrt{D}=\frac{1}{2^\varepsilon}(\alpha_1\sqrt{\mu}+\beta_1\sqrt{\nu})^2.$$

In the case $\chi=2\nu+1$ *the middle element must be of the second category and it is always characterized by*

$$(4)\qquad D=[(D+1)/2]^2-[(D-1)/2]^2.$$

Being given α_2 and β_2, we find first, in virtue of

$$(5)\qquad \begin{aligned}\sqrt{\alpha_2+\beta_2\sqrt{D}}+\sqrt{\alpha_2-\beta_2\sqrt{D}}&=\sqrt{2(\alpha_2+1)},\\ \sqrt{\alpha_2+\beta_2\sqrt{D}}-\sqrt{\alpha_2-\beta_2\sqrt{D}}&=\sqrt{2(\alpha_2-1)},\end{aligned}$$

that

$$(6)\qquad 2\sqrt{\alpha_2+\beta_2\sqrt{D}}=\sqrt{2(\alpha_2+1)}+\sqrt{2(\alpha_2-1)}.$$

When α_2 is odd we must get $\sqrt{2(\alpha_2+1)} = 2\alpha_1\sqrt{\mu}$, $\sqrt{2(\alpha_2-1)} = 2\beta_1\sqrt{\nu}$ giving a solution of $\mu\alpha_1^2 - \nu\beta_1^2 = 1$, and when α_2 is even a solution of $\mu\alpha_1^2 - \nu\beta_1^2 = 2$. And now from (6) we immediately obtain (3). We shall treat as particularly interesting the case $\mu = D$, $\nu = 1$, α_1 and β_1 leading to a solution of $\alpha_1^2 - D\beta_1^2 = -1$. But the solution that we must take from the chain must be that of $(D\beta_1)^2 - D\alpha_1^2 = D$, whence $Q_\nu = D$, and about this element a formation (16′) groups itself for $\mu = D$, $\nu = 1$. With D all $D - n^2 > 0$ $(n = 1 \cdots \tau)$ must also be denominators of reduced fractions belonging to the chain, whence we have the corollary: *The equation $x^2 - Dy^2 = -1$ being soluble, D itself may be expressed in the form $\alpha^2 + \beta^2 - D\gamma^2$ and the number n of such essentially different expressions equals the largest n given by $D - n^2 > 0$.* Conversely, α_1 and β_1 being given so that $\alpha_1^2 - D\beta_1^2 = -1$ we compute

$$\frac{(D-\sqrt{D})(D\beta_1+\alpha_1\sqrt{D})}{D} = (D\beta_1-\alpha_1)+\sqrt{D}(\alpha_1-\beta_1),$$

and $(D\beta_1 - \alpha_1)^2 - D(\alpha_1 - \beta_1)^2 = D - 1$, which is easily verified. Having also $(D\beta_1)^2 - D\alpha_1^2 = D$, and because of the well-known relations*

$$(7) \qquad y_\tau = \beta_\tau y_{\tau-1} - y_{\tau-2}, \qquad z_\tau = \beta_\tau z_{\tau-1} - z_{\tau-2},$$

and the formula (10) of § 1 for $Q_0 = 1$, $(1y_{\tau-1} - P_0 z_{\tau-1})^2 - Dz_{\tau-1}^2 = Q_\tau$ and $D\beta_1(\alpha_1 - \beta_1) - \alpha_1(D\beta_1 - \alpha_1) = -1$,† we can directly calculate x_2 and y_2, solving $x_2^2 - Dy_2 = D - 2^2$ and getting, since $\beta_\tau = 2$,

$$(8) \qquad D\beta_1 = 2(D\beta_1 - \alpha_1) - x_2, \qquad \alpha_1 = 2(\alpha_1 - \beta_1) - y_2.$$

In the same way the following x_i and y_i belonging to the denominators $D - i^2$ may be successively computed, which is rather convenient as the continued fractions themselves have often a very long period χ. It may however be observed that the part of the continued fraction with $\beta_\tau = 2$ may easily be reduced, since

$$(9) \qquad \underbrace{\frac{1|}{|2} - \frac{1|}{|2} - \cdots - \frac{1|}{|2}}_{\nu} = \frac{\nu}{\nu+1},$$

---

* Perron, O.: loc. cit. p. 5.

† This shows that these two solutions are formed from two successive fractions $y_{\tau-1}/z_{\tau-1}$, y_τ/z_τ in the same part of the reduced chain, which can be proved something like Theorem 13, Perron, loc. cit. p. 5.

by a simple induction. In the case of the odd period $\chi = 2\nu + 1$ we must have $D = P_{\nu+1}^2 - Q_{\nu+1}^2$. In view of (6), and as (2) must not be satisfied since that would lead to Q_ν as denominator of a fraction of the first category, the only possibility left will be

$$\alpha_1^2 - D\beta_1^2 = -2. \tag{10}$$

As $Q_\nu = -2$, a negative number, does not appear in the chain, we have to look for the Q_ν actually found in the chain. From (6) we conclude α_2 to be even in order to give (10), and thus D is odd and hence also α_1, β_1. Thus $1 - D \equiv -2$, $D \equiv 3 \text{ mod.} 4$, whence $(D-1)/2$ is odd. Further the ratio of $1^2 - 1^2 D = -(D-1)$ and, since

$$\frac{(\alpha_1 + \beta_1\sqrt{D})(1 + 1\sqrt{D})}{2} = \frac{D\beta_1 + \alpha_1}{2} + \sqrt{D}\frac{\alpha_1 + \beta_1}{2}, \tag{11}$$

$\alpha_1^2 - D\beta_1^2 = -2$ will leave the equation

$$\left(\frac{D\beta_1 + \alpha_1}{2}\right)^2 - D\left(\frac{\alpha_1 + \beta_1}{2}\right)^2 = \frac{D-1}{2}. \tag{12}$$

Supposing $(D-1)/2$ and $(\alpha_1 + \beta_1)$ to have any common factor m, we should find from $D \equiv 1 \text{ mod.}(D-1)/2 \alpha_1^2 - \beta_1^2 \equiv -2$, $2 \equiv 0 \text{ mod.} m$, which for m odd is impossible. On account of the congruence

$$\frac{D\beta_1 + \alpha_1}{2} \equiv \frac{D+1}{2}\,\frac{\alpha_1 + \beta_1}{2} \qquad \text{mod.} \frac{D-1}{2},$$

we get, by (12), $[(D+1)/2]^2 \equiv D \text{ mod.}(D-1)/2$, whence this solution (12) must belong to the congruence-root $\pm(D+1)/2$. Thus the reduced fraction

$$\frac{\frac{D+1}{2} + \sqrt{D}}{\frac{D-1}{2}}$$

must appear in the principal chain and its middle element must therefore be formed from $D = [(D+1)/2]^2 - [(D-1)/2]^2$, as is required by the statement in Theorem 9. Further the formation (17) for $\mu = D$, $\nu = 1$ will reach as far as $D - (2n+1)^2 > 0$. Conversely, having by the continued fraction a solution $\alpha_\nu^2 - D\beta_\nu^2 = (D-1)/2$, it will be a simple matter to find a solution of $x^2 - Dy^2 = -2$. As α_ν, β_ν must belong to the root $\pm(D+1)/2$, we get

$$\alpha_\nu \equiv \pm\frac{D+1}{2}\beta_\nu \qquad \text{mod.}\frac{D-1}{2};$$

hence either $\alpha_\nu + \beta_\nu \equiv 0$ or $\alpha_\nu - \beta_\nu \equiv 0 \bmod.(D-1)/2$. Let us assume the latter case and therefore also $\alpha_\nu - D\beta_\nu \equiv 0 \bmod.(D-1)/2$. We may then form the ratio

$$(13) \qquad \frac{(1+1\sqrt{D})(\alpha_\nu - \sqrt{D}\beta_\nu)}{\frac{D-1}{2}} = \frac{\alpha_\nu - D\beta_\nu}{\frac{D-1}{2}} + \sqrt{D}\frac{\alpha_\nu - \beta_\nu}{\frac{D-1}{2}},$$

and we have in this way gained a solution of $x^2 - Dy^2 = -2$.

Example. $D = 51$.

$$\frac{8+\sqrt{51}}{1} = 16 - \frac{8-\sqrt{51}}{1} = 16 - \frac{1}{\frac{8+\sqrt{51}}{13}},$$

$$\frac{18+\sqrt{51}}{21} = 2 - \frac{24-\sqrt{51}}{21} = 2 - \frac{1}{\frac{24+\sqrt{51}}{25}},$$

$$\frac{8+\sqrt{51}}{13} = 2 - \frac{18-\sqrt{51}}{13} = 2 - \frac{1}{\frac{18+\sqrt{51}}{21}},$$

$$\frac{24+\sqrt{51}}{25} = 2 - \frac{26-\sqrt{51}}{25} = 2 - \frac{1}{\frac{26+\sqrt{51}}{25}} \quad \&c.$$

Period $2\nu + 1 = 7$. We verify in fact $(61 - 8.4)^2 - 51.4^2 = 25$, and by (13) $4.51 - 29 = 7.25$, $4 - 29 = -1.25$ leading to the solution of $7^2 - 51.1^2 = -2$.

§ 5.

In this paragraph we must demonstrate the connection between the chains of the positively and negatively reduced fractions, in order to show their relations to Gauss's classical theory of quadratic forms. We shall first prove this theorem.

THEOREM 1. *From a periodically closed chain of positive type it is always possible to pass over directly to a chain of negative type.*

In all periodic chains, as we have seen, there exist Q_τ with $-P_\tau \equiv \alpha_\tau$ mod. Q_τ and $\alpha_\tau^2 - D < 0$. Putting $(P_\tau + \sqrt{D})/Q_\tau = \mu_\tau + (\sqrt{D} - \alpha_\tau)/Q_\tau$, we see that, either μ_τ or possibly $\nu_\tau > \mu_\tau$ being the greatest integer in $(P_\tau + \sqrt{D})/Q_\tau$, there results

$$\frac{P_\tau + \sqrt{D}}{Q_\tau} = \nu_\tau + \frac{\sqrt{D} - [\alpha_\tau + (\nu_\tau - \mu_\tau)Q_\tau]}{Q_\tau},$$

$$\sqrt{D} - [\alpha_\tau + (\nu_\tau - \mu_\tau)Q_\tau] > 0,$$

since the fraction must be added. In all cases there exists an equality

$$\frac{P_\tau + \sqrt{D}}{Q_\tau} = \nu_\tau + \frac{\sqrt{D} - P_{\tau+1}}{Q_\tau} = \nu_\tau + \cfrac{1}{\cfrac{\sqrt{D} + P_{\tau+1}}{Q_{\tau+1}}},$$

with $P_{\tau+1} > 0$ and $0 < (\sqrt{D} - P_{\tau+1})/Q_\tau < 1$. We can also assume the element $(P_\tau + \sqrt{D})/Q_\tau$ to be chosen so that in $P_\tau^2 - D = Q_{\tau-1} Q_\tau$ we still have $Q_{\tau-1} > Q_\tau$. Combining this with $D - P_{\tau+1}^2 = Q_\tau Q_{\tau+1}$ we have

$$P_\tau^2 - P_{\tau+1}^2 = (P_\tau - P_{\tau+1})(P_\tau + P_{\tau+1}) = Q_\tau (Q_{\tau-1} + Q_{\tau+1}) > Q_\tau^2.$$

If $P_\tau + P_{\tau+1} = Q_\tau$, then $P_\tau - P_{\tau+1} > Q_\tau$, which is impossible. Therefore we must have $P_\tau + P_{\tau+1} \geq 2 Q_\tau$ and $\nu_\tau \geq 2$ and thus $(\sqrt{D} + P_{\tau+1})/Q_\tau = \nu_\tau - (P_\tau - \sqrt{D})/Q_\tau > 1$, whence $(\sqrt{D} + P_{\tau+1})/Q_\tau$ must be a negatively reduced fraction. Concluding further, in the same way as for positively reduced fractions, that they must all exist within the periodic chains, our Theorem 1 is proved. As a corollary we get: *In all periodic chains of positive type there exist elements* $Q_\tau < 2\sqrt{D}$, as can be seen from $0 < (\sqrt{D} - P_{\tau+1})/Q_\tau < 1$, $(\sqrt{D} + P_{\tau+1})/Q_\tau > 1$, giving $Q_\tau < \sqrt{D} + P_{\tau+1} < 2\sqrt{D}$.

THEOREM 2. *The complementary chain of a positively reduced fraction is always connected with the complementary chain of the negatively reduced fraction.*

We have $(P_\rho + \sqrt{D})/Q_\rho = \beta_\rho - (P_{\rho+1} - \sqrt{D})/Q_\rho$, $(P_\rho + \sqrt{D})/Q_\rho = \nu_\rho + (\sqrt{D} - r_\rho)/Q_\rho$, $(\sqrt{D} + r_\rho)/Q_\rho = \delta_\rho + (\sqrt{D} - r_{\rho+1})/Q_\rho$ and thus $P_{\rho+1} + P_\rho \equiv 0$, $P_\rho + r_\rho \equiv 0$, $r_{\rho+1} + r_\rho \equiv 0 \bmod. Q_\rho$, whence $P_{\rho+1} + r_{\rho+1} \equiv 0 \bmod. Q_\rho$, and the theorem is proved.

Two fractions ω_ν and ω_τ between which the relation

$$\omega_\nu = \frac{\alpha\omega_\tau + \beta}{\gamma\omega_\tau + \delta}$$

exists are said, to be properly equivalent when $\alpha\delta - \beta\gamma = +1$ and improperly equivalent when $\alpha\delta - \beta\gamma = -1$.

THEOREM 3. *The same definition of the negatively reduced fractions is valid also for Gauss's reduced forms. Thus each form gives rise to a reduced fraction, and each form can also be represented by a single fraction.*

By $0 < (\sqrt{D} - P_\rho)/Q_\rho < 1$, $(\sqrt{D} + P_\rho)/Q_\rho > 1$ it follows that $\sqrt{D} - P_\rho < Q_\rho < \sqrt{D} + P_\rho$, $0 < P_\rho < \sqrt{D}$, and in this same way Gauss has defined the two reduced forms $(Q_\rho, P_\rho, -Q_{\rho-1})$ and $(-Q_\rho, P_\rho, Q_{\rho-1})$.* Now let $(-Q_\rho, P_\rho,$

---

* Gauss, C. F.: Untersuchungen über Höhere Mathematik (Deutsch v. Maser, H.), 1889, p. 152.

Q_{p-1}) be represented by the reduced fraction $(\sqrt{D}+P_\rho)/-Q_\rho$. Then we have realized a one-to-one correspondence between forms and fractions. It may be observed that proceeding from these fractions we can always form chains leading only to substitutions of proper equivalence (determinant $+1$), as follows

$$\frac{9+\sqrt{105}}{-6} = (-3) - \frac{9-\sqrt{105}}{-6} = (-3) - \cfrac{1}{\cfrac{9+\sqrt{105}}{4}},$$

$$\frac{7+\sqrt{105}}{-14} = (-1) - \frac{7-\sqrt{105}}{-14} = (-1) - \cfrac{1}{\cfrac{7+\sqrt{105}}{4}},$$

$$\frac{9+\sqrt{105}}{4} = 4 - \frac{7-\sqrt{105}}{4} = 4 - \cfrac{1}{\cfrac{7+\sqrt{105}}{-14}},$$

$$\frac{7+\sqrt{105}}{4} = 4 - \frac{9-\sqrt{105}}{4} = 4 - \cfrac{1}{\cfrac{9+\sqrt{105}}{-6}}.$$

Following up the parallel between Gauss's theory and this theory of periodic chains, we shall find that Gauss's method of forming adjacent forms* is exactly the same as our method of forming a chain, which the reader will have no difficulty in seeing from a numerical example, say $D = 79$.† Thus, since these reduced fractions with negative denominator can always be brought over into reduced chains of positive type, we have achieved the fundamental.

THEOREM 4. *The number of Gauss's classes equals the number of positively periodic chains.*

Further we shall prove this important theorem.

THEOREM 5. *If the fundamental unity η has a negative norm ($N(\eta) = -1$), all periodically closed chains of negative type will have an odd period, but, if $N(\eta) = +1$, all periods will be even.*

Proceeding from the negatively reduced fraction $(\sqrt{D}+P_\rho)/Q_\rho$ and forming the chain as above, this chain will be closed with $(\sqrt{D}+P_\rho)/Q_\rho$ in all chains of even period but with $(\sqrt{D}+P_\rho)/-Q_\rho$ in the case of an odd period. In this latter case we take by Theorem 7 § 1 a solution of $x^2 - Dy^2 = -(2^\varepsilon)^2$ ($\varepsilon = 0$ or 1). Thus we have either $N(\eta) = -1$ or $\alpha^2 - D\beta^2 = -4$, $N(\eta) = +1$, which is impossible. For we find $\lambda^3 = [(\alpha+\beta\sqrt{D})/2]^3 = [(\alpha-\beta)/2 + \beta(\sqrt{D}+1)/2]^3 = m + n\sqrt{D}$ (m, n integers) from $D \equiv 5$

---

* Gauss, C. F., loc. cit., p. 118.

† Gauss, C. F., loc. cit., p. 159.

mod. 8. Hence $N(\lambda^3) = -1$ contrary to $N(\eta) = +1$. Thus, with only one of the periods odd there must result $N(\eta) = -1$. But then all forms $(-a, b, c)$ and $(a, b, -c)$ are properly equivalent[*] and hence also the two reduced fractions $(P_\rho + \sqrt{D})/Q_\rho$ and $(P_\rho + \sqrt{D})/-Q_\rho$, whence they stay in a reduced chain of odd period. Thus for $N(\eta) = -1$ all closed chains have necessarily an odd period and for $N(\eta) = +1$ they have an even period. Hence we get the corollary: *If* $N(\eta) = +1$, *the number of positively reduced chains is twice that of the negatively reduced chain, but if* $N(\eta) = -1$ *these numbers are equal.*

THEOREM 6. *Ambiguous classes of forms*[†] *correspond to symmetrically closed chains with elements* $(\sqrt{D} + P_\nu)/Q_\nu$ *of the first category*

$$D \equiv 0 \bmod. Q_\nu/2^\epsilon.$$

This follows immediately from Gauss's definition of ambiguous forms and from (5) § 1, it being observed that for $D > 0$ every ambiguous class of forms certainly contains two ambiguous forms.[‡] But given the form $(Q_\rho, P_\rho, -Q_{\rho-1})$ with $Q_{\rho-1} = Q_\rho$, there exists a corresponding symmetrical chain without any element of the first category and therefore only of the second if $N(\eta) = +1$.

Example. $D = 34$, $34 = 3^2 + 5^2$.

$$\frac{\sqrt{34}+3}{5} = 1 + \frac{\sqrt{34}-2}{5} = 1 + \cfrac{1}{\cfrac{\sqrt{34}+2}{6}},$$

$$\frac{\sqrt{34}+5}{3} = 3 + \frac{\sqrt{34}-4}{3} = 3 + \cfrac{1}{\cfrac{\sqrt{34}+4}{6}},$$

$$\frac{\sqrt{34}+2}{6} = 1 + \frac{\sqrt{34}-4}{6} = 1 + \cfrac{1}{\cfrac{\sqrt{34}+4}{3}},$$

$$\frac{\sqrt{34}+4}{6} = 1 + \frac{\sqrt{34}-2}{6} = 1 + \cfrac{1}{\cfrac{\sqrt{34}+2}{5}},$$

$$\frac{\sqrt{34}+3}{3} = 3 + \frac{\sqrt{34}-5}{3} = 3 + \cfrac{1}{\cfrac{\sqrt{34}+5}{3}}$$

$$\frac{\sqrt{34}+2}{5} = 1 + \frac{\sqrt{34}-3}{5} = 1 + \cfrac{1}{\cfrac{\sqrt{34}+3}{5}}.$$

---

[*] Sommer, J.: Vorlesungen über Zahlentheorie, p. 203.

[†] Gauss, C. F., loc. cit., p. 125.

[‡] Gauss, C. F., loc. cit., p. 215 and art. 194.

Then we remark that symmetric chains of positive type must always pass over into symmetric chains of negative type. But for $N(\eta) = +1$ there exists a symmetrical negative chain having two elements of the second category and to this chain there cannot correspond any symmetric positive chain, since the two derived forms are improperly but never properly equivalent.

Example. $D = 34$.

$$\frac{7+\sqrt{34}}{3} = 4 + \frac{\sqrt{34}-5}{3} = 4 + \cfrac{1}{\cfrac{\sqrt{34}+5}{3}},$$

$$\frac{12+\sqrt{34}}{11} = 2 - \frac{10-\sqrt{34}}{11} = 2 - \cfrac{1}{\cfrac{10+\sqrt{34}}{6}},$$

$$\frac{7+\sqrt{34}}{3} = 5 - \frac{8-\sqrt{34}}{3} = 5 - \cfrac{1}{\cfrac{8+\sqrt{34}}{10}},$$

$$\frac{10+\sqrt{34}}{6} = 3 - \frac{8-\sqrt{34}}{6} = 3 - \cfrac{1}{\cfrac{8+\sqrt{34}}{5}},$$

$$\frac{8+\sqrt{34}}{10} = 2 - \frac{12-\sqrt{34}}{10} = 5 - \cfrac{1}{\cfrac{12+\sqrt{34}}{11}},$$

$$\frac{8+\sqrt{34}}{5} = 3 - \frac{7-\sqrt{34}}{5} = 3 - \cfrac{1}{\cfrac{7+\sqrt{34}}{3}}.$$

By Theorem 4 it must be possible to establish a one-to-one correspondence between positively reduced fractions appertaining to $\sqrt{D}$ and quadratic forms $(Q_\rho, -P_\rho, Q_{\rho-1})$ of determinant D. This is accomplished by choosing as correspondents $(P_\rho + \sqrt{D})/Q_\rho$ and $(Q_\rho, -P_\rho, Q_{\rho-1})$. But the definition of a reduced form will now be distinct from that of Gauss, the form $(Q_\rho, -P_\rho, Q_{\rho-1})$ being reduced for values Q_ρ, P_ρ from $P_\rho - \sqrt{D} < Q_\rho < P_\rho + \sqrt{D}$, $\sqrt{D} < P_\rho < D$.

Thus for example we see from

$$\frac{-15+\sqrt{105}}{5} = 0 - \frac{15-\sqrt{105}}{5} = 0 - \cfrac{1}{\cfrac{15+\sqrt{105}}{24}},$$

that the substitution of determinant $+1$, $x = 0x' - 1y'$, $y = 1x' + 0y'$, transforms $5x^2 + 30xy + 24y^2$ into the reduced form $24x'^2 - 30x'y' + 5y'^2$, and from

$$\frac{15+\sqrt{105}}{-5} = (-4) - \frac{\sqrt{105}-5}{5} = (-4) - \frac{1}{\frac{\sqrt{105}+5}{16}},$$

$$\frac{5+\sqrt{105}}{16} = \quad 1 - \frac{11-\sqrt{105}}{16} = \quad 1 - \frac{1}{\frac{11+\sqrt{105}}{1}},$$

that the substitution of proper equivalence $x = -5x' + 4y'$, $y = 1x' - 1y'$ transforms $-5x^2 - 30xy - 24y^2$ into the reduced form $x'^2 - 22x'y' + 16y'^2$. In this connection we may also point out the disadvantage that in this new definition of reduced forms ambiguous classes can exist without reduced ambiguous forms, as we see from 8) § 2. But a theory of quadratic forms, based on these reduced forms, may be established, which will retain at least the principal features of Gauss's theory, as may be seen from what has been developed in this paragraph.

Having established a one-to-one correspondence between certain fractions of the domain $K(\sqrt{D})$ and quadratic forms of the determinant D, we shall be able to compound fractions analogously to the well-known Gaussian composition of forms. This is readily done. Hence we may speak of species of compositions of chains analogously to that of form classes. On the researches of Gauss* we base the following: Let (a, b, c) and (a', b', c') be two quadratic forms with the same D; a, $2b$, c having the greatest common factor m and a', $2b'$, c' the greatest common factor m', and let m, m' be relatively prime, then the two forms may be compounded into the third form (A, B, C),

$$\frac{a\,a'}{d^2} = A,$$

d being the greatest common factor of a, a', $b + b'$ and

$$\frac{aB}{d} \equiv \frac{a\,b'}{d}, \quad \frac{a'B}{d} \equiv \frac{a'b}{d}, \quad \frac{b+b'}{d}B \equiv \frac{b\,b'+D}{d} \quad \text{mod.} A.$$

It may be observed that the third congruence follows from the preceding two; for assume as given $b \equiv D \,\text{mod.}\, a$, $b' \equiv D \,\text{mod.}\, a'$ and try to determine $x^2 \equiv D$ mod. $a\,a'/d$, then we obtain from $y \equiv b \,\text{mod.}\, a$, $y \equiv b' \,\text{mod.}\, a'/d$, $y^2 \equiv D$ mod. $a\,a'/d$ the following congruences, $(y-b)(y-b') = y^2 + b\,b' - y\,(b+b')$ $\equiv 0 \,\text{mod.}\, dA$, $(y^2 + b\,b')/d \equiv (D + b\,b')/d \equiv y(b+b') \,\text{mod.}\, A$, that is, for $y \equiv B \,\text{mod.}\, A$ just the Gaussian formulas, and they must determine B since a/d, a'/d, $(b+b')/d$ have no common factor†. If already $(b+b')/d$ and A are relatively prime, B will be directly obtained by the last congruence. Thus we have settled a close connection between the Gaussian classes of quadratic forms and the positively closed chains; and there also exist a very close connection between the negatively closed chains and the ideal classes. I shall in a future paper prove the following general theorem: *If* $D \equiv 2, 3 \bmod. 4$ *the number of ideal classes equals that of the negatively closed chains.*

---

* Gauss, C. F.,: loc. cit. p. 249 and 251.

† Bachmann, I.: Grundlehren der Neueren Zahlentheorie, 1907, p. 249.

For $D \equiv 1 \bmod. 4$ we have two types of fractions, as of quadratic forms, a well-known fact, one type characterized by odd as well as even denominators, the other by only even denominators. Hence: *If* $D \equiv 1 \bmod. 4$ *the number of ideal classes equals that of the negatively closed chains of the second type. If* $D \equiv 1 \bmod. 8$ *or* $D \equiv 5 \bmod. 8$ *with* $x^2 - Dy^2 = \pm 4$ *soluble in integers* x, y, *the number of the closed chains of either type is the same, but this is no longer true if* $D \equiv 5 \bmod. 8$, $x^2 - Dy^2 \neq \pm 4$. We verify in fact for $D = 105 \equiv 1 \bmod. 8$ four negatively reduced fractions, since $N(\eta) = +1$, $\eta = 41 + 4\sqrt{105}$, and two chains of either type, whence the number of ideal classes will be 2 with the more extensive equivalence, otherwise 4.

§ 6.

In this paragraph we shall briefly deal with chains formed in the case of negative D and shall demonstrate their connection with quadratic forms. Let us form the chain

$$(1') \qquad P_\nu^2 + D = Q_{\nu-1} Q_\nu, \qquad P_{\nu+1}^2 + D = Q_\nu Q_{\nu+1}, \text{ \&c.}$$

or the formation of fractions

$$(1'') \qquad \frac{P_\nu + i\sqrt{D}}{Q_\nu} = \beta_\nu - \frac{P_{\nu+1} - i\sqrt{D}}{Q_\nu} = \beta_\nu - \cfrac{1}{\cfrac{P_{\nu+1} + i\sqrt{D}}{Q_{\nu+1}}},$$

where $P_{\nu+1}$ is always the absolutely least remainder of $P_\nu \bmod. Q_\nu$ and therefore $|P_{\tau+1}| \leq Q_\tau/2$. For $Q_\tau \geq D$ we have

$$P_{\tau+1}^2 + D = Q_\tau Q_{\tau+1} \leq (Q_\tau/2)^2 + Q_\tau < Q_\tau^2,$$

which is true with $Q_\tau > 4/3$, and hence for all $Q_\tau \geq 2$ and generally $Q_\tau \geq Q_{\tau+1}$. This process is to be continued till the absolutely least remainder of $P_{\rho+1} \bmod. Q_{\rho+1}$ is $P_{\rho+1}$ itself. Then we have either $Q_\rho > Q_{\rho+1}$, and we shall illustrate below how we may bring this "reduced link" into connection with Gauss's reduced forms $(\pm Q_{\rho+1}, P_{\rho+1}, \pm Q_\rho)$, or we have $Q_\rho < Q_{\rho+1}$ and

$$(2) \qquad P_{\rho+1}^2 + D = Q_\rho Q_{\rho+1}, \qquad P_{\rho+2}^2 + D = Q_{\rho+1} Q_{\rho+2},$$

with $P_{\rho+2} = -P_{\rho+1}$, $Q_{\rho+2} = Q_\rho$, and from the latter link we take the reduced forms $(\pm Q_{\rho+2}, P_{\rho+2}, \pm Q_{\rho+1})$. For we get in both cases $(Q_{\rho+i+1}/2)^2 + D > P^2_{\rho+i+1} + D = Q_{\rho+1} Q_{\rho+i+1} > Q^2_{\rho+i+1}$, $Q_{\rho+i+1} \leq \sqrt{4D/3}$, $|P_{\rho+i+1}| \leq Q_{\rho+i+1}/2$, $Q_{\rho+i+1} \leq Q_{\rho+i}$ $(i = 0 \text{ or } 1)$, which are Gauss's conditions of a reduced form.*

From the reduced link we form

$$\text{(3)}\qquad P^2_{\rho+2} + D = (s Q_{\rho+1} + |P_{\rho+1}|)^2 + D = Q_{\rho+1} Q_{\rho+2}$$
$$= Q_{\rho+1}(s^2 Q_{\rho+1} + 2s|P_{\rho+1}| + Q_\rho),$$

with $s \gtrless 0$. For $s > 0$ we have

$$2(s Q_{\rho+1} + |P_{\rho+1}|) \leq s^2 Q_{\rho+1} + 2s|P_{\rho+1}| + Q_\rho,$$
$$0 < (s-1)^2 Q_{\rho+1} + 2(s-1)|P_{\rho+1}| + Q_\rho - Q_{\rho+1},$$

since $Q_\rho \geq Q_{\rho+1}$, and hence $|P_{\rho+2}|$ is the absolutely least remainder mod. $Q_{\rho+2}$. Then according to the rule of chain formation the next link must be formed as in (2), and after this we get once more the known reduced link. For $s = -s'$ we find by

$$0 \leq (s'-1)[(s'-1)Q_{\rho+1} - 2|P_{\rho+1}|] + Q_\rho - Q_{\rho+1},$$

that $P_{\rho+2}$ is still the absolutely least remainder mod. $Q_{\rho+2}$. It remains therefore to test whether in (3) any reduced link may be constructed. For $s > 0$ this is obviously impossible, but for $s = -s'$ we hav eto satisfy $s' Q_{\rho+1} - |P_{\rho\mp1}| \leq Q_{\rho+1}/2$, which is realized only for $s' = 1$, $|P_{\rho+1}| = Q_{\rho+1}/2$. From

$$(Q_{\rho+1}/2)^2 + D = Q_\rho Q_{\rho+1},$$
$$(Q_{\rho+1}/2)^2 + D = Q_{\rho+1} Q_\rho,$$
$$(Q_{\rho+1}/2) + D = Q_\rho Q_{\rho+1},$$

or

$$\frac{Q_{\rho+1}/2 + i\sqrt{D}}{Q_{\rho+1}} = 1 - \frac{Q_{\rho+1}/2 - i\sqrt{D}}{Q_{\rho+1}} = 1 - \cfrac{1}{\cfrac{Q_{\rho+1}/2 + i\sqrt{D}}{Q_\rho}},$$

$$\frac{Q_{\rho+1}/2 + i\sqrt{D}}{Q_\rho} = 0 - \frac{-Q_{\rho+1}/2 - i\sqrt{D}}{Q_\rho} = 0 - \cfrac{1}{\cfrac{-Q_{\rho+1}/2 + i\sqrt{D}}{Q_{\rho+1}}},$$

we have the well-known fact that the substitution of determinant $+1$, $x = x' - y'$, $y = y'$, transforms the ambiguous form $(Q_{\rho+1}, +Q_{\rho+1}/2, Q_\rho)$ into

---

* Gauss, C. F.: loc. cit., p. 136.

$(Q_{\rho+1}, -Q_{\rho+1}/2, Q_\rho)$. Having $Q_{\rho+1} = Q_\rho$ in (2), we readily find the proper equivalence of the forms $(Q_\rho, +P_{\rho+1}, Q_\rho)$, $(Q_\rho, -P_{\rho+1}, Q_\rho)$, and hence the problem of the equivalence of forms with negative D is solved entirely as Gauss* has already solved it. Thus for negative D also, we can settle the problem of equivalence almost as before in the case of positive D, by the formation of chains like (1′) and (1″), and then we can also determine the number of the classes of forms.

Example. From

$$20^2+5 = 15.27, \quad 7^2+5 = 27.2, \quad 1^2+5 = 2.3, \quad 1^2+5 = 3.2,$$

or

$$\frac{20+i\sqrt{5}}{27} = 1 - \frac{7-i\sqrt{5}}{27} = 1 - \cfrac{1}{\cfrac{7+i\sqrt{5}}{2}},$$

$$\frac{7+i\sqrt{5}}{2} = 4 - \frac{1-i\sqrt{5}}{2} = 4 - \cfrac{1}{\cfrac{1+i\sqrt{5}}{3}},$$

$$\frac{1+i\sqrt{5}}{3} = 0 - \frac{-1-i\sqrt{5}}{3} = 0 - \cfrac{1}{\cfrac{-1+i\sqrt{5}}{2}},$$

we find the substitution of determinant $+1$, $x = 1x' - 3y'$, $y = -1x' - 4y'$, transforms $(27x^2 + 40xy + 15y^2)$ into the ambiguous form $(2x'^2 - 2x'y' + 3y'^2)$. Lastly, we can give the solution of an indeterminate equation

$$x^2 - Dy^2 = A$$

in integers x and y, based on the preceding theorem in § 3 with some slight modifications, in addition to solving the more general problem of expressing a given integer A in the form (a, b, c).

LUND, SWEDEN.
October, 1921.

---

* Gauss, C. F.: loc. cit. p. 138.